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Let us briefly review some definitions.

Definition 10.1

A *presheaf* $\mathcal{O}_X$ of sets (rings/modules) on a topological space X is a set (ring/-module) $\mathcal{O}_X(U)$ for every open set $U \subset X$, together with the following data:

1. For every inclusion of open sets $V \subset U$ in X we have a *restriction map* $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ which is a map of sets (rings/modules).
2. For open sets $W \subset V \subset U$ in X , $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.
3. For all open sets $U \subset X$, the restriction map ρ_{UU} is the identity.

A presheaf is a *sheaf* if we have an additional two axioms:

1. Identity axiom: If $U \subset X$ is an open set, $f, g \in \mathcal{O}_X(U)$, and $\{V_i : i \in I\}$ is an open covering of U , such that for all $i \in I$,

$$\rho_{UV_i}(f) = \rho_{UV_i}(g),$$

then $f = g$.

1. Gluability axiom: If U is an open set in X , and $\{V_i : i \in I\}$ is an open covering of U , and for every V_i we have $f_i \in \mathcal{O}_X(V_i)$ such that for all $i, j \in I$,

$$\rho_{V_i V_i \cap V_j}(f_i) = \rho_{V_j V_i \cap V_j}(f_j),$$

then there exists an $f \in \mathcal{O}_X(U)$ such that $\rho_{UV_i}(f) = f_i$ for all $i \in I$.

A *morphism of sheaves* $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set $U \subset X$, such that for every inclusion of open sets $V \subset U$ in X the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \Big\downarrow \rho^{\mathcal{F}}_{U\,V} & & \Big\downarrow \rho^{\mathcal{G}}_{U\,V} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V). \end{array}$$

For $x \in X$, the *stalk* at x is the set of equivalence classes $\{[(f,U)] : f \in \mathcal{O}_X(U), x \in U\}$ under the equivalence relation $(f,U) \sim (g,V)$ if there exists an open $x \in W \subset U \cap V$ such that $\rho_{UW}(f) = \rho_{VW}(g)$. We denote the germ $[(f,U)]$ as f_x .

Presheaves can be made into sheaves by sheafification. This can be expressed by a universal property, and hence is unique up to unique isomorphism.

Let $\pi : X \to Y$ be a continuous map of topological spaces, and $\mathcal{O}_X$ a sheaf on X . The *direct image* $\pi_*(\mathcal{O}_X)$ of $\mathcal{O}_X$ is the presheaf on Y given by, for $V \subset Y$ open, $\pi_*\mathcal{O}_X(V) = \mathcal{O}_X(\pi^{-1}(V))$. This is in fact a sheaf. There is also a pullback of a sheaf, $\pi^*\mathcal{O}_Y$.

Definition 10.2

A ringed space is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X . A morphism of ringed spaces $(\pi, \pi_\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map of topological spaces $\pi : X \to Y$, and a morphism of sheaves on Y , $\pi_\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$. Equivalently, we could define $\pi_\#$ to be a morphism of sheaves on X , $\pi^*(\mathcal{O}_Y) \to \mathcal{O}_X$. A *locally ringed space* is a ringed space such that ring of germs at each point is local. A *morphism of locally ringed spaces* is a morphism of ringed spaces, with the additional requirement that it takes the maximal ideal of the germ in X to the maximal ideal of the germ in Y for every $x \in X$. Morphisms of locally ringed spaces induce maps of stalks. That is, if $x \in X, y = \pi(x)$, there is induced morphism of rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, $[(f,U)] \mapsto [(\pi_\#(U)(f), \pi^{-1}(U))]$ where $\pi_\#(U)(f) \in \pi_*\mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}(U))$ as needed.

A commutative ring with unity can be made into a locally ringed space using the Spec functor. Let R be a ring. As a topological space, let $Spec = \{p : p \text{ is a prime ideal of } R\}$. Define maps

$$Z(-) : \{\text{ideals in } R\} \longrightarrow \{\text{ sets in } Spec\}, Z(S) = \{p \in Spec | S \subset p\},$$

$$I(-) : \{\text{sets in } Spec\} \longrightarrow \{\text{ideals in } R\}, I(K) = \{f \in R | f \in p \text{ for all } p \in K\}.$$

The *Zariski topology* on $Spec$ is defined by saying $Z(S)$ is closed for every $S \subset R$

. The following lemma lists some well-known facts about these maps.

Lemma 10.1

Let R be a ring, $J \subset R$ an ideal in R , and $K \subset \mathit{Spec}$ a closed set.

1. $J \subset I(Z(J))$.
2. $K = Z(I(K))$.
3. $I(Z(J)) = \sqrt{J}$, where $\sqrt{J} = \{f |$ there exists $n \in \mathbb{N}$ such that $f^n \in J\}$.

The topology on Spec has as an open basis $D(f) = \{p \in \mathit{Spec} | f \notin p\}$ for all $f \in R$. We think of elements in R as functions on Spec , where the value of f at p is the projection of f in R/p . However, because of nilpotents (which are precisely elements in $\bigcap_{p \in \mathit{Spec}} p$), functions may not be determined by their values at points. In the particular case of the ring $R = K[X_1, \ldots, X_n]$, where K is an algebraically closed field, functions (elements in R) are determined by their values on the spectrum, and moreover, they are determined by their value at the maximal ideals of R , which are in one to one correspondence with elements in K^n .

The ringed space $\mathit{Spec} = (X, \mathcal{O}_R)$ is $X = \mathit{Spec}$ as a topological space, together with the sheaf on the base of distinguished open set (sets of the form $D(f), f \in R$), where $\mathcal{O}_R(D(f))$ is the localization of R at the set of all elements $g \in R$: $D(f) \subset D(g)$. This in fact defines a sheaf on a base. An affine scheme is a ringed space which is isomorphic to $(\mathit{Spec}, \mathcal{O}_R)$ for some ring R . A scheme is a ringed space $(X, \mathcal{O}_X)$ which can be covered by open sets such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. If $\phi : R \to S$ is a morphism of commutative rings, then it induces a morphisms of affine sheaves $\mathit{Spec} \to \mathit{Spec}$. We want morphisms of schemes to locally look like the morphisms that arise in this way. One can define morphisms of schemes like this, but equivalently, morphisms of locally ringed spaces coincide with them, which gives an alternative definition.

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1.1 Text

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