Let us now explore the consequences of Widom’s assumption on the critical exponents of a system, again on a magnetic one for concreteness.

1 Exponent beta

Since $M = \partial f / \partial H$, deriving both sides of Widom’s assumption with respect to $h$ \cite{1} we get:

$$\lambda^{p_2} \frac{\partial}{\partial h} f_s(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda \frac{\partial}{\partial h} f_s(t, h)$$

and thus:

$$\lambda^{p_2} M(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda M(t, h)$$

In order to determine $\beta$, we set $h = 0$ so that this becomes:

$$\lambda^{p_2 - 1} M(\lambda^{p_1} t, 0) = M(t, 0)$$

and using the properties of generalized homogeneous functions, we set $\lambda = -t^{-1/p_1}$ to get:

$$M(t, 0) = (-t)^{\frac{1-p_2}{p_1}} M(-1, 0)$$

By definition of the $\beta$ critical exponent, we have:

$$\beta = \frac{1 - p_2}{p_1}$$

2 Exponent delta

We can determine the exponent $\delta$ setting $t = 0$:
\[ M(0, h) = \lambda^{p_2-1} M(0, \lambda^{p_2} h) \]

Now, using again the same property of generalized homogeneous functions we set \( \lambda = h^{-1/p_2} \) and get:

\[ M(0, h) = h^{\frac{1-p_2}{p_2}} M(0, 1) \]

so that:

\[ \delta = \frac{p_2}{1 - p_2} \]

Now we can also express \( p_1 \) \( p_2 \) \( \beta \) \( \delta \):

\[ p_1 = \frac{1}{\beta(\delta + 1)} \quad p_2 = \frac{\delta}{\delta + 1} \]

from which we see that the gap exponent is:

\[ \Delta = \frac{p_2}{p_1} = \beta \delta \]

## 3 Exponent gamma

In order to obtain the magnetic susceptibility, we derive twice the expression of Widom’s assumption with respect to \( h \), to get:

\[ \lambda^{2p_2} \chi_T(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda \chi_T(t, h) \]

The exponent \( \gamma \) describes the behaviour of \( \chi_T \) for \( t \to 0 \) when no external field is present. What we can now see is that the scaling hypothesis leads to the equality of the exponents for \( t \to 0^+ \) and \( t \to 0^- \) (which we just assumed for simplicity in Critical exponents and universality).

Setting \( h = 0 \) and \( \lambda = (-t)^{-1/p_1} \) we get:

\[ \chi_T(t, 0) = (-t)^{-\frac{2p_2-1}{p_1}} \chi_T(-1, 0) \]

and if we call \( \gamma^- \) the critical exponent for \( t \to 0^- \), we see that:

\[ \gamma^- = \frac{2p_2 - 1}{p_1} \]

In order to compute the exponent \( \gamma^+ \) that describes the behaviour of \( \chi_T \) for \( t \to 0^+ \), we set \( \lambda = t^{-1/p_1} \), so that the susceptibility becomes:

\[ \chi_T(t, 0) = t^{-\frac{2p_2-1}{p_1}} \chi_T(1, 0) \]

so that indeed:
5 Griffiths and Rushbrooke’s equalities

\[ \gamma^+ = \frac{2p_2 - 1}{p_1} \]

We therefore see explicitly that:

\[ \gamma^+ = \gamma^- = \gamma = \frac{2p_2 - 1}{p_1} \]

which using the previous expressions of \( p_1 \), \( p_2 \), \( \beta \) \( \delta \) leads to:

\[ \gamma = \beta(\delta - 1) \]

4 Exponent alpha

In order to determine the behaviour of the specific heat (at constant external field) near the critical point, we derive the expression of Widom’s assumption twice with respect to the temperature, so that:

\[ \lambda^{2p_1} C_H(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda C_H(t, h) \]

Setting \( h = 0 \) and \( \lambda = (-t)^{-1/p_1} \):

\[ C_H(t, 0) = (-t)^{-\left(2-\frac{1}{p_1}\right)} C_H(-1, 0) \]

so:

\[ \alpha^- = 2 - \frac{1}{p_1} \]

Again, we can see that this exponent is equal to the one that we get for \( t \to 0^+ \); in fact, setting \( \lambda = t^{-1/p_1} \):

\[ C_H(t, 0) = t^{-\left(2-\frac{1}{p_1}\right)} C_H(-1, 0) \]

so that indeed \( \alpha^+ = \alpha^- \). Therefore:

\[ \alpha = 2 - \frac{1}{p_1} \]

5 Griffiths and Rushbrooke’s equalities

If we now substitute

\[ p_1 = \frac{1}{\beta(\delta + 1)} \]

into

\[ \alpha = 2 - \frac{1}{p_1} \]
An alternative expression for the scaling hypothesis

We can re-express Widom’s assumption in another fashion often used in literature. If we set $\lambda = t^{-1/p_1}$, then:

$$f_s(1, t^{-p_2/p_1} h) = t^{-1/p_1} f_s(t, h)$$

From

$$\Delta = \beta \delta$$

and

$$\alpha = 2 - \frac{1}{p_1}$$

we can rewrite this as:

$$f_s(t, h) = t^{2-\alpha} f_s \left( 1, \frac{h}{t^{\Delta}} \right)$$

which is the most used form of the scaling hypothesis in statistical mechanics. As we can notice, we have not considered the critical exponents $\eta$ and $\nu$; this will be done shortly in Kadanoff’s scaling and correlation lengths.

1. We should in principle derive with respect to $H$, but since $h \propto \beta H$, the $\beta$ factors simplify on both sides.
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