Complex Analysis (Intermediate Level)

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Chapter 1

Complex numbers

1.1 Introduction

In this first section we recall some basic concepts about complex numbers. A deep knowledge of these notions is essential for the comprehension of the following topics of the course.

The need of extending the real field \( \mathbb{R} \) has its origins in the impossibility of finding a solution of particular equations, such as:

\[
x^2 + 1 = 0 ,
\]

for which it is clear that \( \nexists x \in \mathbb{R} : x^2 + 1 = 0 \).

We define the complex field \( \mathbb{C} \), so that we can solve equations such as the previous one.

\( \mathbb{C} \) is defined through the extension of the well-known properties of \( \mathbb{R} \).

We remember that \( \mathbb{R} \) is a vector field if it is equipped with the following operations:

1. Addition

\[
\forall x, y \in \mathbb{R} \text{ we have that } x + y \in \mathbb{R} ,
\]

1. Multiplication

\[
\forall x, y \in \mathbb{R} \text{ we have that } xy \in \mathbb{R} ,
\]

defined in an axiomatic way, in order to satisfy the following properties:

- Commutative property: \( \forall x, y \in \mathbb{R} \) we have that:

\[
\begin{align*}
x + y &= y + x , \\
x y &= y x .
\end{align*}
\]
• Associative property: \( \forall x, y, z \in \mathbb{R} \) we have that:

\[
x + (y + z) = (x + y) + z, \\
x(yz) = (xy)z.
\]

• Distributive property: \( \forall x, y, z \in \mathbb{R} \) we have that:

\[
x(y + z) = xy + xz.
\]

• Identity element: \( \forall x \in \mathbb{R} \) we have that:

\[
x + 0 = 0 + x = x, \\
1x = x1 = x.
\]

• Inverse element (addition):

\[
\forall x \in \mathbb{R} \ \exists x \in \mathbb{R} : \ x + (-x) = (-x) + x = 0.
\]

• Inverse element (multiplication):

\[
\forall x \in \mathbb{R} \setminus \{0\} \ \exists \frac{1}{x} \in \mathbb{R} : \ x \left( \frac{1}{x} \right) = \left( \frac{1}{x} \right) x = 1.
\]

We define the complex field \( \mathbb{C} \) as the set of the pairs \( (a, b) \) with \( a, b \in \mathbb{R} \) equipped with the properties:

1. Equality:

\[
\forall (a, b) \& (c, d) \in \mathbb{C} \ we \ have \ that \ (a, b) = (c, d) \iff \begin{cases} a = c \\ b = d \end{cases},
\]

1. Addition:

\[
\forall (a, b) \& (c, d) \in \mathbb{C} \ we \ have \ that \ (a, b) + (c, d) = (a + c, b + d),
\]

1. Multiplication:

\[
\forall (a, b) \& (c, d) \in \mathbb{C} \ we \ have \ that \ (a, b)(c, d) = (ac - bd, ad + bc).
\]

we see that if we define \( \mathbb{C}_0 \equiv \{(a, b) \in \mathbb{C} : b = 0\} \) we have that:
\[ C_0 \subset C; \]
\[ C_0 \text{ is a field, with the operations and properties inherited by } C; \]
\[ C_0 \text{ is isomorphic to } \mathbb{R}, \text{ that is:} \]

\[ \exists f : \mathbb{R} \to C \ x \mapsto f(x) = (x, 0) \text{ is an isomorphism.} \]

Moreover, it can be proved that the function:

\[ g : \mathbb{R}^2 \to C \text{ definita come: } g(x, y) = (x, y) \forall x, y \in \mathbb{R} \]

is an isomorphism, that is \( C \) is isomorphic to \( \mathbb{R}^2 \).

### 1.2 Algebraic form of complex numbers

We start this section by recalling the concept of *imaginary unit*.

**Definition**

We define the imaginary unit as the element \( (0, 1) \in C \), and we call it \( i \).

Now we compute the multiplication, according to the previous definition given in \( C \), of \( (0, 1) \) by itself:

\[ (0, 1)(0, 1) = (-1, 0) \equiv -1, \]

where we have used the isomorphism between \( C \) and \( \mathbb{R}^2 \).

Therefore, the imaginary unit is such that \( i^2 = -1 \): so, it solves the equation \( x^2 + 1 = 0 \).

Analogously, we refer to the complex number \( (0, -1) \) as \(-i\); we remark that this is another solution of the previous equation.

So, we can write: \( i = \sqrt{-1} \).

Through \( i \) we can give an algebraic form of complex number, that allows us to simplify operations between them. In particular:

\[ \forall (a, b) \in C \text{ we have that } (a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1). \]

In this way, a complex number \( (a, b) \) can be written, by the previous isomorphism and the definition of imaginary unit, in the following way:

**Definition**

A complex number \( (a, b) \) can be represented in the form:

\[ (a, b) = a + ib. \]

This is the algebraic form.
Definition

Given $z = a + ib$, we call, respectively, real part and imaginary part of $z$ the two real numbers $a$ and $b$ and we refer to them as:

$$Re(z) \equiv a, \quad Im(z) \equiv b.$$  

Definition

Given $z = a + ib \in \mathbb{C}$, we define the complex conjugate of $z$ the complex number $\bar{z} = a - ib$, that has the same real part and the opposite imaginary part. For all $z \in \mathbb{C}$ the following properties are valid:

- $z + \bar{z} = 2 Re(z)$,
- $z - \bar{z} = 2i Im(z)$,
- $z\bar{z} = a^2 + b^2 = |z|^2$.

The following definition of module is an extension of its usual concept in $\mathbb{R}$.

Definition

We define the module of $z$ as the number $r = |z| \in \mathbb{R}$:

$$r = \sqrt{(Re(z))^2 + (Im(z))^2}.$$  

So, it derives naturally the following:

Theorem

$(\mathbb{C}, | \cdot |)$ is a metric space.

1.3 Polar form and graphical representation of complex numbers

Being that $\mathbb{C} \cong \mathbb{R}^2$, we can think at representing complex numbers on a plane similar to the Cartesian one. The complex plane is known as Argand-Gauss plane.

This plane as the real part of the chosen complex number as first coordinate, and the imaginary part as the second one.

From this graphical representation it is easily derived the polar form of a complex number. In particular, if $z = a + ib \in \mathbb{C}$, we set:

$$\begin{cases} 
a = Re(z) = r \cos \theta \\
b = Im(z) = r \sin \theta \end{cases}$$  

where

$$\begin{cases} 
r^2 = a^2 + b^2 = |z|^2 \\
\theta = \arctan \left( \frac{b}{a} \right) + 2n\pi, \ n \in \mathbb{Z} \end{cases}$$
Actually, in order to well-define a complex number in polar coordinates we have to choose an interval of definition for the angle $\theta$, for instance let $\theta$ vary in $[-\pi; \pi]$. The angle $\theta$ is called the argument of the complex number $z$.

**Theorem** (Euler’s Formula)

We see that every complex number in trigonometric form can be rewritten as:

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}.$$ 

We provide two alternative proofs of the Euler’s Formula.

**Proof**

We consider the series $\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$.

This is absolutely convergent since the series of the modules:

$$\sum_{n=0}^{\infty} \left| \frac{(i\theta)^n}{n!} \right| = \sum_{n=0}^{\infty} \frac{|\theta^n|}{n!}$$

converges in $\mathbb{R}$. Moreover, it is known that the sum of the series is $e^{i\theta}$, so:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}.$$ 

Now, we can rewrite the series by separating the terms in an even and odd position as follows:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{(2n+1)}}{(2n+1)!} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}.$$ 

The two series are both power convergent series (it has been proven in the course of Multivariate Calculus). In fact $\sin x$ and $\cos x$ are two examples of functions of class $C^\infty$ that satisfy the right conditions on the growth of the $n$-th order derivative, so that we can write them as sum of power series. In particular, we have that:

$$\cos(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!},$$

$$\sin(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}.$$ 

And so we achieve that:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$
Proof

Given the exponential function $e^{i\theta}$, we expect that $\exists A, B : \mathbb{R} \to \mathbb{C} : e^{i\theta} = A(\theta) + iB(\theta)$.

We want to prove that $A(\theta) = \cos \theta$ and $B(\theta) = \sin \theta$.

We take the derivative of $e^{i\theta}$ with respect to $\theta$:

$$ie^{i\theta} = A'(\theta) + iB'(\theta).$$

By taking a second derivative, we have that:

$$-e^{i\theta} = A''(\theta) + iB''(\theta).$$

We compare the two equations we have with $e^{i\theta} = A(\theta) + iB(\theta)$ and we obtain the following system of differential equations:

$$\begin{cases}
A''(\theta) = -A(\theta) \\
B''(\theta) = -B(\theta)
\end{cases}$$

With the initial conditions $A(0) = 1$, $A'(0) = 0$, $B(0) = 0$, $B'(0) = 1$.

By solving these equations we have that:

$$A(\theta) = \cos \theta \quad \text{and} \quad B(\theta) = \sin \theta.$$

Example 1 For the imaginary unit $z = i$, we have that:

$$z = i = (0, 1) \Rightarrow z = e^{i\frac{\pi}{2}}.$$

1.4 DeMoivre’s Formula and roots of a complex number

The next theorem provides an important tool to calculate the $n$-th power of a complex number:

**Theorem** (DeMoivre’s Formula)

\[ \forall z \in \mathbb{C}, \, n \in \mathbb{Z} \text{ we have that:} \]

\[ z^n = (ze^{i\theta})^n = r^n(\cos(n\theta) + i\sin(n\theta)) \].

The computation of the $n$-th roots of a complex number is a bit more difficult. That is, given $z \in \mathbb{C}$ we are asking if $\exists w \in \mathbb{C} : w^n = z$ or, analogously, $w = z^{\frac{1}{n}}$. The number $w$ is called the $n$-th root of $z$. 
By considering the polar form of a complex number we have:

\[ z = re^{i\theta} = re^{i(\theta + 2k\pi)}, \quad k \in \mathbb{Z} \]

from what it follows that:

\[ w = z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \quad \forall k = 0, 1, \ldots, n - 1. \]

So, we have that every complex number admits \( n \) distinct roots.

**Example**

We compute the \( n \)-th roots of the unity. So, we consider \( z = 1 \), and we want to find \( w = 1^{\frac{1}{n}} \).

\[ z = 1 = e^{i2k\pi} \Rightarrow w = e^{i\frac{2k\pi}{n}}, \quad \text{that is:} \]

\[
\begin{align*}
  k &= 0 \quad & w_0 &= 1 \\
  k &= 1 \quad & w_1 &= e^{i\frac{2\pi}{n}} \\
  k &= 2 \quad & w_2 &= e^{i\frac{4\pi}{n}} = w_1^2 \\
  \vdots & & \vdots \\
  k &= n - 1 \quad & w_{n-1} &= e^{i\frac{2\pi(n-1)}{n}} = w_1^{n-1}
\end{align*}
\]

From this example we see that the \( n \) distinct roots of a complex number represent, in the plane \((\text{Re}(z), \text{Im}(z))\) defined via the isomorphism between \( \mathbb{C} \) and \( \mathbb{R}^2 \), the \( n \) vertices of a regular polygon.
Chapter 2

Functions in the complex plane

2.1 Introduction

In the following chapters, we will see that, in order to define complex functions and their features, we can extend some properties of real functions; but, in some cases the mere extension will not be enough.

We start with the following:
Consider \( \mathcal{D} \subset \mathbb{C} \). A complex-valued function is a function:

\[
f : \mathcal{D} \rightarrow \mathbb{C}.
\]

We see that \( \forall z = x + iy \in \mathcal{D} \) the function \( f \) can be decomposed as:

\[
f(z) = u(z) + iv(z),
\]

where the functions \( u, v \) are real-valued functions. In particular, we define:

\[
u = \text{Re}(f) \quad \text{&} \quad v = \text{Im}(f).
\]

Alternatively, we can use the isomorphism between \( \mathbb{C} \) and \( \mathbb{R}^2 \) and look at \( u \) and \( v \) as two variables real functions:

\[
f(z) = f(x + iy) = u(x, y) + iv(x, y).
\]

In fact, by calling \( \tilde{\mathcal{D}} \subset \mathbb{R}^2 \), we have that: \( u, v : \tilde{\mathcal{D}} \rightarrow \mathbb{R} \).

Now we define some complex functions as extensions of known real functions:

**Trigonometric Functions**: We recall that, thanks to the Euler’s formula, we can write:

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2},
\]
So, the natural extension to the complex field is:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

Analogously, we have that:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

by setting $z = x + iy$ we see that:

$$\sin(z) = \sin(x + iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}.$$

We underline that $|e^{\pm ix}| = e^{\pm ix}(e^{\pm ix})^{*} = e^{\pm ix}e^{-\pm ix} = 1$. All the complex exponentials of the form $e^{if(x)}$, where $f(x)$ can be also a constant function, are called phases and have unitary modulus.

The following properties (simple to prove) are valid:

- $\cos^2 z + \sin^2 z = 1$
- $\cos(iy) = \cosh y$
- $\sin(iy) = i \sinh y$
- $\cos(z) = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
- $\sin(z) = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

**Logarithm function** in $\mathbb{C}$: In order to define the complex logarithm, we use the polar form of complex numbers. So, we have:

$$\log(z) = \log(|z|, e^{i\theta}) = \log(|z|, e^{i\text{Arg}(z)}) = \log(|z|) + i\text{Arg}(z).$$

Actually, we have to keep in mind that complex numbers are defined with a certain ambiguity in their polar form, so we have that:

$$\log(z) = \log(|z|) + i(\text{Arg}(z) + 2k\pi), \ k \in \mathbb{Z}.$$

The following properties are true:

- $\log(-1) = i(\pi + 2k\pi)$
- $\log(i) = i(\frac{\pi}{2} + 2k\pi)$
- $\forall z_1, z_2 \in \mathbb{C},$ we have that $\log(z_1z_2) = \log(z_1) + \log(z_2) + 2k\pi i$
- $\forall z_1, z_2 \in \mathbb{C},$ we have that $\log(\frac{z_1}{z_2}) = \log(z_1) - \log(z_2) + 2k\pi i$. 

---
2.2 Continuity and differentiability in the complex case

We provide the definitions of continuous and differentiable function in \( \mathbb{C} \). We will see that it is not possible to simply extend the properties of real functions.

**Definition**

We consider \( D \subset \mathbb{C} \) and \( z_0 = x_0 + iy_0 \in D \). We say that \( f : D \rightarrow \mathbb{C} \) is continuous at the point \( z_0 \) if:

\[
\forall \epsilon > 0 \ \exists \delta > 0 : \forall z \in D, |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon.
\]

By recalling that \( \forall z \in \mathbb{C} \) we have that \( z = x + iy \) and so that \( f(z) = f(x + iy) = u(x,y) + iv(x,y) \), we state and prove the following

**Theorem**

We consider \( D \subset \mathbb{C} \) and \( \tilde{D} \subset \mathbb{R}^2 \). Also \( z_0 = x_0 + iy_0 \in \tilde{D} \). The function \( f : D \rightarrow \mathbb{C} \) is continuous at the point \( z_0 \) if and only if the functions \( u(x,y) \) & \( v(x,y) \) as defined in (1.1) are continuous, in the notion of the continuity of functions defined on \( \mathbb{R}^n \) made during analysis II, in point \( (x_0,y_0) \in \tilde{D} \).

**Proof**

Let \( u, v \) be continuous at the point \( (x_0,y_0) \in \tilde{D} \). We have that:

\[
|f(z) - f(z_0)| = |u(x,y) + iv(x,y) - u(x_0, y_0) - iv(x_0, y_0)| = \\
= |u(x,y) - u(x_0, y_0) + i(v(x,y) - v(x_0, y_0))| = \\
= \sqrt{(u(x,y) - u(x_0, y_0))^2 + (v(x,y) - v(x_0, y_0))^2} \\
\leq |u(x,y) - u(x_0, y_0)| + |v(x,y) - v(x_0, y_0)|.
\]

We used the definition of \( f(z) \) and the elementar inequality:

\[
\sqrt{\sum x_i^2} \leq \sum |x_i|.
\]

In an analogous way:

\[
|z - z_0| \leq |x - x_0| + |y - y_0|.
\]

We know, by hypothesis, that \( u, v \) are continuous at \( (x_0,y_0) \), that is:

\[
\forall \epsilon_1 \exists \delta_1 : \forall (x,y) \in \tilde{D} \ | (x,y) - (x_0,y_0) | \leq \delta_1 \Rightarrow |u(x,y) - u(x_0, y_0)| \leq \epsilon_1,
\]

\[
\forall \epsilon_2 \exists \delta_2 : \forall (x,y) \in \tilde{D} \ | (x,y) - (x_0,y_0) | \leq \delta_2 \Rightarrow |v(x,y) - v(x_0, y_0)| \leq \epsilon_2.
\]

Therefore we have that, called \( \epsilon = \epsilon_1 + \epsilon_2 \) & \( \delta = \delta_1 + \delta_2 \):
\[ \forall z \in \mathcal{D}, \ |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \]

Vice versa let \( f \) be continuous at \( z_0 = x_0 + iy_0 \in \mathcal{D} \). From the definition of continuity for complex function we have that:

\[ \forall \epsilon > 0 \exists \delta > 0 : \forall z \in \mathcal{D}, \ |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \]

We see that:

\[
|u(x, y) - u(x_0, y_0)| \leq |f(z) - f(z_0)| \quad \& \quad |v(x, y) - v(x_0, y_0)| \leq |f(z) - f(z - z_0)|,
\]

\[
|x - x_0| \leq |z - z_0| \quad \& \quad |y - y_0| \leq |z - z_0|.
\]

So, given \( \epsilon_1 \) and \( \epsilon_2 \), we obtain:

\[
\exists \delta_1 : \forall (x, y) \in \mathcal{D} , |(x, y) - (x_0, y_0)| \leq \delta_1 \Rightarrow |u(x, y) - u(x_0, y_0)| \leq \epsilon_1,
\]

\[
\exists \delta_2 : \forall (x, y) \in \mathcal{D} , |(x, y) - (x_0, y_0)| \leq \delta_2 \Rightarrow |v(x, y) - v(x_0, y_0)| \leq \epsilon_2.
\]

That is, \( u, v \) are continuous at \((x_0, y_0) \in \mathcal{D}\).

Now, we introduce the definition of \textit{differentiable functions} in \( \mathbb{C} \); it will be clear that, while the definition of continuity is simply an extension of the known one for the functions in \( \mathbb{R} \), the notion of differentiability is more complicated that the pure extension of a known fact.

**Definition**

Consider \( \mathcal{D} \subset \mathbb{C} \) and \( z_0 = x_0 + iy_0 \in \mathcal{D} \). We say that \( f : \mathcal{D} \to \mathbb{C} \) is differentiable at the point \( z_0 \) if the following limit exists and is unique:

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).
\]

The value of this limit is the \textit{derivative of the function \( f \) at the point \( z_0 \)}.

**Remark 1:** Stating that the limit is unique is equivalent to state that it must exists and be the same for every path followed to reach \( z_0 \).

Now, we provide some necessary conditions for differentiability of a complex function. We will see that, in some cases, these conditions can become sufficient. We repeat that the limit \( \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \), must exist for every path used to reach \( z_0 \); we study two simple paths: those for which we arrive at \( z_0 \) along parallel axis to those of \( \text{Re}(z) \) & \( \text{Im}(z) \) in the complex plane:

- Along the axis parallel to \( \text{Re}(z) \): A generic \( z \in \mathbb{C} \) who belongs to this axis has the form \( z = x + iy_0 \). Therefore we have that:

\[
\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x - x_0}.
\]
By taking the limit for \( z \to z_0 \) we have that:

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x - x_0} = \\
= \lim_{x \to x_0} \left[ \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \right] = \\
= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0).
\]

- Along the axis parallel to \( Im(z) \): A generic \( z \in \mathbb{C} \) who belongs to this axis has the form \( z = x_0 + iy \). Therefore, as before, we have that:

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0).
\]

We remind that if \( f(z) \) is differentiable at \( z_0 \) the limit at \( z_0 \) must exist and be unique. The two found limits, if \( f(z) \) is differentiable at \( z_0 \), must be the same. We give the following

**Definition**

The condition:

\[
\begin{cases}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{cases}
\]

are called the **Cauchy-Riemann equations**.

For what said about the differentiability of a function at \( z_0 \), these equations are necessary conditions for the differentiability of a function at \( z_0 \). But, they are not sufficient: if the function is differentiable the two limits computed by getting closer along the real and the imaginary axis must be the same because the uniqueness of the limit along each path is valid. If the function is not differentiable, the two limits can coincide even if the limit is not unique along each path. It can exist a path that denies the existence of a limit and so making the function not differentiable, even if it satisfies the Cauchy-Riemann equations.

**Theorem**

We consider \( \mathcal{D} \subset \mathbb{C} \) and \( z_0 = x_0 + iy_0 \in \mathcal{D} \). If \( f : \mathcal{D} \to \mathbb{C} \) and it is derivable at the point \( z_0 \), then it satisfies the Cauchy-Riemann condition

**Remark 2:**

If Cauchy-Riemann equations are satisfied, then:

\[
\frac{\partial f}{\partial \bar{z}}(x_0, y_0) = 0.
\]

**Proof**
We notice that:

\[
\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad \text{from Cauchy-Riemann equations it follows that:}
\]

\[
= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = -i \frac{\partial f}{\partial y}.
\]

So, the Cauchy-Riemann equations are equivalent to:

\[
\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.
\]

We compute \( \frac{\partial f}{\partial \bar{z}} \) as derivative of a composed function, since we can write:

\[
x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = i \left( \frac{z - \bar{z}}{2} \right),
\]

so we have:

\[
\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y} = 0.
\]

We have three necessary and equivalent conditions for the differentiability of a composed function in \( z_0 \in \mathbb{C} \):

\[
\begin{cases}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \\
\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial \bar{z}}(x_0, y_0) = 0.
\end{cases}
\]

**Definition**

We consider \( D \subset \mathbb{C} \) and \( z_0 = x_0 + iy_0 \in D \). If \( f : D \to \mathbb{C} \) is differentiable at the point \( z_0 \), we say that \( f \) is holomorphic in \( z_0 \). If \( \Omega \subset D \) we say that \( f \) is holomorphic in \( \Omega \) if it has this property at every points \( z \in \Omega \).

As already said, in some cases the necessary conditions for the differentiability of \( f \) can also be sufficient. In fact, more rigorously:

**Theorem**

Given \( D \subset \mathbb{C} \) and \( f : D \to \mathbb{C} \), we consider \( \Omega \subset D \). If partial derivatives of \( f \) exist and are continuous in \( \Omega \), so Cauchy-Riemann equations are necessary and sufficient conditions for the differentiability of \( f \) in \( \Omega \). Equivalently, we can say that \( f \) is holomorphic in \( \Omega \) if and only if it satisfies the Cauchy-Riemann equations.

We do not provide a proof of this fact, but we consider the following examples:
Chapter 2. Functions in the complex plane

1. \( f(z) = f(x, y) = x \), that is the complex function that \( z \mapsto \text{Re}(z) \).
   We see that \( \frac{\partial f}{\partial x} = 1 \neq 0 = \frac{\partial f}{\partial y} \). Cauchy-Riemann equations are not satisfied, so we conclude that \( f \) is not holomorphic.

2. \( f(z) = e^z = e^x \cos(y) + ie^x \sin(y) \). By using Cauchy-Riemann equations:

\[
\begin{align*}
  e^x \cos(y) &= e^x \cos(y) \\
  -e^x \sin(y) &= e^x \sin(y)
\end{align*}
\]

By seeing that it is true \( \forall x, y \in \mathbb{R} \) we conclude that \( f \) is holomorphic.

1. \( f(z) = \sqrt{|xy|}, \ z = x + iy \). First of all we see that \( v(x, y) \equiv 0 \), while \( u(x, y) = \sqrt{|xy|} = \text{Re}(f) \equiv f \). We want to apply the Cauchy-Riemann equations, from which we want to obtain that \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \).

\[
\begin{align*}
  \frac{\partial u}{\partial x} &= \frac{1}{2} \sqrt{\frac{y}{x}} |\text{sgn}(x)|, \\
  \frac{\partial u}{\partial y} &= \frac{1}{2} \sqrt{\frac{x}{y}} |\text{sgn}(y)|.
\end{align*}
\]

First of all, we see that both the partial derivatives, if computed respectively at \( y = 0 \) & \( x = 0 \) are zero.

By computing the incremental limit of \( f \) along a line passing through the origin of the axis in the plane \( \mathbb{C} \) we have:

\[
\frac{f(z) - f(0)}{z - 0} = \frac{\sqrt{r^2 |\cos(\theta)\sin(\theta)|}}{r(\cos(\theta) + i\sin(\theta))} = \frac{\sqrt{\cos(\theta)\sin(\theta)}}{\cos(\theta) + i\sin(\theta)}
\]

By taking the limit for \( z \to 0 \) it is evident that it depends on \( \theta \), that is the chosen path to reach \( z_0 = 0 \).
So, \( f'(0) \) cannot exist.

**Remark 3:** We will see a relationship between the solutions of the Laplace equation \( \nabla^2 f = 0 \) and the class of the holomorphic function in \( \mathbb{C} \).

We consider Laplace equation in cartesian coordinates:

\[
\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.
\]

Let be \( g(z) \) a generic holomorphic function on \( \Omega \) and be \( x = \text{Re}(z) \) & \( y = \text{Im}(z) \). We can state that:

\[
\frac{\partial g}{\partial x} = -i \frac{\partial g}{\partial y}.
\]

If \( g \) is partially differentiable up to the second order, we can derive with respect to \( x \) the previous equation, and we have:
\[
\frac{\partial^2 g}{\partial x^2} = -i \frac{\partial^2 g}{\partial x \partial y} = -i \frac{\partial}{\partial y} \left( \frac{\partial g}{\partial x} \right) = -\frac{\partial^2 g}{\partial y^2}, \text{ that is:}
\]
\[
\nabla^2 g = 0.
\]

The class of the holomorphic function in \( \mathbb{C} \) coincides with the class of the solutions of the Laplace equation in two dimensions.

We do not go beyond in the theory of differentiability in \( \mathbb{C} \), because the possibility to derive to orders greater than the second (in particular, to derive a function infinite times) will be strictly related to the theory of the integration in \( \mathbb{C} \).

### 2.3 Integrability in the complex case

We introduce the theory of integrable functions in \( \mathbb{C} \). From the integrability of a function it will follow that it is differentiable infinite times, where it is holomorphic.

We will not consider generic integrals, but integrals along paths.

Therefore, we have to define curves and paths in \( \mathbb{C} \).

**Definition**

A curve in \( \mathbb{C} \) is a function \( \gamma : [a, b] \to \mathbb{C} \) such that:

\[
\forall t \in [a, b], \ \gamma(t) \in \mathbb{C}.
\]

**Definition**

The set \( \gamma^* = \gamma([a, b]) = \{ \gamma(t) \in \mathbb{C} : t \in [a, b] \} \) is defined as the support of the curve \( \gamma \). The points \( \gamma(a) \) and \( \gamma(b) \) are respectively the first and the second extremum of the curve. If \( \gamma(a) = \gamma(b) \) we say that the curve is closed.

**Definition**

A curve \( \gamma \) in \( \mathbb{C} \) is a path if it is piecewise differentiable.

Now, we can give a meaning to the expression integrate along a path. The definition we will give is the same given in the course of Multivariable Calculus for the elements of an oriented path. All the remarks given in the case of a multivariable function follow: among them, we have the invariance of the integral with respect to the chosen oriented path. In the following, we can take \( [a, b] \equiv [0, 1] \).

**Definition**

Let \( \gamma : [a, b] \to \mathbb{C} \) be a path and \( f : D \subset \mathbb{C} \to \mathbb{C} \) a complex-valued function. We define integral of \( f \) along the path \( \gamma \) the quantity:

\[
\int_{\gamma} f(z)dz = \int_{a}^{b} f(\gamma(t))\gamma'(t)dt.
\]
**Definition**

We define the length of the curve (or the path) $\gamma$ as the quantity:

$$L(\gamma) \equiv \int_0^1 |\gamma'(t)| \, dt.$$  

We have that:

$$|\int_\gamma f(z)dz| \leq ML(\gamma), \text{ where } M \equiv \max\{f(z) : z \in \gamma^*\}.$$

An important concept, that distinguishes the integral along paths in $\mathbb{C}$ from the one in $\mathbb{R}^2$ is the **index**. In particular, we have the following:

**Definition**

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed path and $\Omega = \mathbb{C} \setminus \gamma^*$. \forall z \in \Omega we define **index of z with respect to $\gamma$** the quantity:

$$\text{Ind}_\gamma(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\xi}{\xi - z} \equiv \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)\gamma(t) - z}{\gamma(t) - z} dt.$$

In this definition we go through $\gamma$ in a counter-clockwise way. We set $\text{Ind}_\gamma(\infty) = 0$. We can see that every paths $\gamma$ in $\mathbb{C}$ satisfy an important property, analogous to the one given by the **Jordan Theorem** for simple and closed curves in $\mathbb{R}^2$. In fact, we have that $\gamma^*$ always divides $\mathbb{C}$ in two connected components, one bounded and the other unbounded. There is an important relation between the function $\text{Ind}_\gamma(z)$ and the condition of holomorphism, as given by the following

**Theorem**

$\text{Ind}_\gamma(z)$ is an holomorphic function in $\mathbb{C}$. Moreover, we have that $\text{Ind}_\gamma : \Omega \subset \mathbb{C} \to \mathbb{Z}$, that is $\text{Ind}_\gamma$ is an integer valued function and has constant value on each connected component in which $\mathbb{C}$ is splitted by $\gamma$. On the exterior of $\gamma$ the index is always zero.

As for the definition of interior and exterior, we follow the **Jordan Theorem** for simple and closed curves in $\mathbb{R}^2$. We can refer to books of Multivariate Calculus about this topic.

We provide a proof of the previous theorem in a simple case:

**Proof**

Let $\xi \in \mathbb{C}$ and $\gamma$ be a circumference with centre $z_0$ and radius $r$. So, we have that $\gamma(t) = re^{it} + z_0, \ t \in [0, 2\pi]$.

We obtain that:

$$\text{Ind}_\gamma(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\xi}{\xi - z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)\gamma(t) - z}{\gamma(t) - z} dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = 1.$$
We remind the definitions of connected, convex & simply connected set.

**Definition**

We say that $\Omega \subset \mathbb{C}$ is connected if:
\[ \forall z_1, z_2 \in \Omega \text{ we have that } z(t) = (1 - t)z_1 + tz_2 \subset \Omega. \]

In general, for the other two definitions, we refer to textbooks of Multivariate Calculus, where one can find rigorous definitions for $\Omega \subset \mathbb{R}^n, n \geq 2$ that can be easily extended to $\mathbb{C}$.

We state two of the most important results in complex analysis, given the notion:

\[ \hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}. \]

**Theorem**

We consider $\Omega \subset \hat{\mathbb{C}}$ and $f$ an holomorphic function in $\Omega$. If $\gamma$ is a closed path in $\Omega$ such that $\text{Ind}_\gamma(z) = 0, \forall z \in \hat{\mathbb{C}} \setminus \Omega$, then:

\[ \int_{\gamma} f(z)dz = 0. \]

Equivalently, we have that:

Let $\Omega \subset \hat{\mathbb{C}}$ be a simply connected set (and so $\hat{\mathbb{C}} \setminus \Omega$ is connected) and $f$ be a holomorphic function in $\Omega$. If $\gamma$ is a closed path in $\Omega$, then:

\[ \int_{\gamma} f(z)dz = 0. \]

We see that the theorem provides a necessary condition for the integral along the path to be zero, in fact we could have cases in which the curve has a non-zero index and so we could not say anything about the value of the integral a priori.

An important corollary of Cauchy theorem gives conditions for the comparison between integrals of the same holomorphic $f$ along two different closed paths in $\Omega$.

**Theorem**

We consider $\Omega \subset \hat{\mathbb{C}}$ and $f$ an holomorphic function in $\Omega$. Let $\gamma_1$ & $\gamma_2$ be two closed paths in $\Omega$ such that $\text{Ind}_{\gamma_1}(z) = \text{Ind}_{\gamma_2}(z), \forall z \in \hat{\mathbb{C}} \setminus \Omega$, then:

\[ \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz. \]

**Proof**

Let $f$ be holomorphic in $\Omega$ and $\gamma, \lambda$ be two closed paths.

We link $\gamma$ and $\lambda$ with two line segments $\pm c$ with a distance $\epsilon$ between them.
It is not possible to have a closed path with zero index with respect to every \( z \in \mathbb{C} \setminus \Omega \). Let such \( \Gamma \) given by: \( \Gamma = \gamma \cup c \cup -\lambda \cup -c \).

By applying Cauchy’s theorem to the function \( f \) considered:

\[
0 = \int_{\Gamma} f(z)dz = \int_{\gamma} f(z)dz + \int_{c} f(z)dz + \int_{(-\lambda)} f(z)dz + \int_{(-c)} f(z)dz = \\
= \int_{\gamma} f(z)dz + \int_{c} f(z)dz - \int_{(-\lambda)} f(z)dz - \int_{(-c)} f(z)dz = \\
= \int_{\gamma} f(z)dz - \int_{\lambda} f(z)dz.
\]

It follows that: \( \int_{\gamma} f(z)dz = \int_{\lambda} f(z)dz \).
Chapter 3

Computation of integrals
Chapter 4

Power Series Background

4.1 Uniform Convergence

Consider the sequence of functions $u_n(x) = x^n$, on the unit interval $J = [0, 1]$. For each fixed $x \in J$, the sequence $u_n(x)$ is convergent:

$$u_n(x) \to u(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$
as $n \to \infty$.

For each fixed $n$, $u_n(x)$ is continuous, but the limiting function $u(x)$ is not. This is an unfortunate state of affairs which is remedied by the introduction of.

This is a stronger notion of convergence for a sequence of functions, sufficient to guarantee, in particular, that continuity is preserved in the passage to limits.

Let $\Omega$ be any set (you may think that it is a subset of $\mathbb{C}$ or $\mathbb{R}$) and let $u_n : \Omega \to \mathbb{C}$ be sequence of functions defined in $\Omega$.

**Definition 1**

The sequence $u_n$ converges uniformly to the function $u : \Omega \to \mathbb{C}$ if: given $\varepsilon > 0$, there exists $N_0$ (depending upon $\varepsilon$) such that

$$n > N_0 \implies |u_n(z) - u(z)| < \varepsilon \text{ for all } z \in \Omega.$$

**Remark 2**

An equivalent formulation of uniform convergence is

$$n > N_0 \implies \sup_{z \in \Omega} |u_n(z) - u(z)| < \varepsilon.$$

If $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is any function, we say that $f$ is continuous at $z_0$ in $\Omega$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$z \in \Omega \text{ and } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon.$$
Then $f$ is said to be continuous in $\Omega$ if it is continuous at every point of $\Omega$.

Now we show that 'the uniform limit of a sequence of continuous functions is again continuous'.

**Theorem 1**

Let $u_n : \Omega \to \mathbb{C}$ be continuous for each $n$ and suppose that $u_n \to u$ uniformly in $\Omega$ as $n \to \infty$. Then $u$ is continuous in $\Omega$.

**Proof**

Pick $z_0 \in \Omega$. We need to estimate

$$|u(z) - u(z_0)|$$

for all $z \in \Omega$ close to $z_0$.

For this, use

$$|u(z) - u(z_0)| = |u(z) - u_n(z) + u_n(z) - u_n(z_0) + u_n(z_0) - u(z_0)|$$

$$\leq |u(z) - u_n(z)| + |u_n(z) - u_n(z_0)| + |u_n(z_0) - u(z_0)|.$$

Given $\varepsilon > 0$, by uniform convergence, there exists $N_0$ so that

$$n > N_0 \implies \sup_{z \in \Omega} |u_n(z) - u(z)| < \varepsilon/3.$$

Substituting this into Uniform convergence, we obtain

$$|u(z) - u(z_0)| = \varepsilon/3 + |u_n(z) - u_n(z_0)| + \varepsilon/3.$$

Now $n > N_0$, and find, by continuity of $u_n$ at $z_0$, a $\delta > 0$ such that

$$|u_n(z) - u_n(z_0)| < \varepsilon/3 \text{ if } z \in \Omega \text{ and } |z - z_0| < \delta.$$

Combining Uniform convergence and Uniform convergence we see that for this same $\delta$,

$$|u(z) - u(z_0)| < \varepsilon \text{ for all } z \in \Omega, |z - z_0| < \delta.$$

which proves that $u$ is continuous.

**Example 3**

We can see that the example we started with fails to be uniformly convergent. Indeed, with $u_n$ and $u$ as in Uniform convergence, if $x < 1$, we have $|u_n(x) - u(x)| = x^n$ and this tends to $1$ as $x \to 1$. Thus $\sup_{x \in J} |u_n(x) - u(x)| = 1$ and so $u_n$ does not converge uniformly to $u$ in this case.
4.1.1 Uniformly convergent series

As usual, we pass from uniformly convergent sequences to uniformly convergent series via partial sums. That is, if $f_n : \Omega \to \mathbb{C}$ is a sequence of functions, then we say that the series $\sum_{n=0}^{\infty} f_n$ is uniformly convergent in $\Omega$ if the corresponding sequence of partial sums

$$F_n = \sum_{j=0}^{n} f_j$$

is uniformly convergent in $\Omega$. There is a for uniform convergence of sums, with the following statement:

**Theorem 2**

Let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions. The series $\sum f_n$ converges uniformly in $\Omega$ if and only if:

Given any $\varepsilon > 0$, there exists $N_0$ depending upon $\varepsilon$, such that for all $n > m > N_0$, we have

$$\sup_{z \in \Omega} \left| \sum_{j=m}^{n} f_j(z) \right| < \varepsilon.$$

Finally, we state the Weierstrass $M$-test, which is a very effective way of checking uniform convergence of many of the series which arise in complex analysis.

**Theorem 3**

Let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions. Suppose that there exist $M_n$ such that

$$\sup_{z \in \Omega} |f_n(z)| \leq M_n$$

and also

$$\sum_{n=1}^{\infty} M_n < \infty.$$ 

Then the series $\sum f_n$ converges absolutely and uniformly in $\Omega$. In particular, if each of the $f_n$ is continuous, then so is $\sum_{n=1}^{\infty} f_n$.

**Proof**

Use the Cauchy criterion. By the triangle inequality,

$$|\sum_{j=m}^{n} f_j(z)| \leq \sum_{j=m}^{n} |f_j(z)| \leq \sum_{j=m}^{n} M_j$$
for every $z \in \Omega$. The Cauchy convergence criterion for the convergent series $\sum M_n$ states that, given $\varepsilon > 0$, there exists $N_0$ such that $n > m > N_0$ implies $\sum_{j=m}^{n} M_j < \varepsilon$. Substituting this into Uniformly convergent series we have

\[
\sum_{j=m}^{n} |f_j(z)| < \varepsilon \text{ for } n > m > N_0 \text{ and all } z \in \Omega
\]

as required for the use of Theorem Uniformly convergent series to get absolute and uniform convergence of $\sum f_n$. The continuity statement follows from Theorem Uniform convergence.

### 4.2 Radius of convergence

Let

\[
\sum_{n=0}^{\infty} a_n z^n
\]

be a power series centred at $z = 0$. Here the $a_n$, the of the power series are fixed complex numbers.

**Proposition 4**

There exists $r$, $0 \leq r \leq \infty$, such that Radius of convergence is absolutely convergent if $|z| < r$ and divergent if $|z| > r$. The convergence is moreover on any smaller closed disc $K = \{ z : |z| \leq r_1 \}$, $(r_1 < r)$.

**Proof**

For any $s \geq 0$ let

\[
M(s) = \sup\{|a_n|s^n : n = 0, 1, 2, \ldots\}
\]

where we allow the possibility $M(s) = \infty$ if the sequence $\{a_n|s^n\}$ is unbounded. We note that if $s < t$, then $M(s) \leq M(t)$ (where this may have to be interpreted as $\infty \leq \infty$). To show this, suppose for example that $M(t) < \infty$. Then

\[
|a_n|s^n = |a_n|(s/t)^n \leq M(t)(s/t)^n \leq M(t).
\]

Thus every element of the sequence $\{a_n|s^n\}$ is $\leq M(t)$ and so the same is true of their sup $M(s)$. A similar argument shows that if $M(s) = \infty$ and $s < t$, then also $M(t) = \infty$.

Since $M(0) = |a_0| < \infty$, it makes sense to ask for the largest $s$ with $M(s) < \infty$. In other words, define

\[
r = \sup\{s : M(s) < +\infty\}.
\]
The claim is that \( r \) has the required properties of the Proposition. (Note that \( M(r) \) can be either finite or \( +\infty \), depending on the particular power series.)

If \( |z| < r \), then there exists \( r_1 \) with \( |z| < \rho < r \). We have seen that \( M(\rho) \) is finite, and so

\[
|a_n||z^n| = |a_n|\rho^n(|z|/\rho)^n \leq M(\rho)(|z|/\rho)^n.
\]

The absolute convergence follows by comparison with the geometric series \( M(\rho) \sum (|z|/\rho)^n \) which is convergent because \( |z| < \rho \).

If \( |z| > r \), then \( M(|z|) = +\infty \) so that the sequence \((|a_n||z|^n)\) is unbounded. Thus the terms in Radius of convergence don’t tend to zero and the partial sums cannot converge.

Finally, let \( K = \{|z| \leq r_1\} \) be a closed subdisc. Let \( \rho \) be chosen so that \( r_1 < \rho < r \). Then we can run the above argument uniformly for all \( |z| < r_1 \) (even \( \leq r_1 \)),

\[
|a_n||z^n| = |a_n|\rho^n(|z|/\rho)^n \leq M(\rho)(|z|/\rho)^n \leq M(\rho)(r_1/\rho)^n
\]

if \( |z| \leq r_1 \). Uniform convergence on \( K \) is equivalent to the following version of Cauchy’s criterion:

Given \( \varepsilon > 0 \), there exists \( N \) (depending upon \( \varepsilon \)) such that if \( m > n > N \), then \( |\sum_{j=n}^{m} a_j z^j| < \varepsilon \) for all \( z \in K \).

This is satisfied in our case for \( K = \{|z| \leq r_1\} \) because

\[
\left| \sum_{j=n}^{m} a_j z^j \right| \leq \sum_{j=n}^{m} |a_j||z|^j \leq M(\rho) \sum_{j=n}^{m} (r_1/\rho)^j \leq \frac{(r_1/\rho)^n}{1 - (r_1/\rho)}
\]

and this tends to zero as \( n \to \infty \) if \( r_1 < \rho \). (In the last inequality in Radius of convergence we have bounded a finite geometric progression by its sum to infinity. This is a device that is often handy.)

**Remark 5**

It follows from the definition that if Radius of convergence has radius of convergence \( r \), then so do the related series

\[
\sum_{n=N}^{\infty} a_n z^n \text{ and } \sum_{n=N}^{\infty} a_n z^{n+k}
\]

where \( N \geq 0 \) is an integer, \( k \) is an integer, and we assume \( N + k \geq 0 \) if \( k \) is negative.

Thus the radius of convergence is unaffected by throwing away any finite number of terms in the sum, or by multiplying by a fixed power of \( z \).

**Remark 6**
Consider the three series
\[
\sum_{n=0}^{\infty} z^n, \\
\sum_{n=0}^{\infty} nz^n, \\
\sum_{n=0}^{\infty} \frac{z^n}{n^2}.
\]
In each case the radius of convergence \( r = 1 \) (exercise). In the case of Radiu
of convergence and Radius of convergence \( M(1) \), defined in Radius of conver
cence, is finite, in fact equal to 1. In the case of Radius of convergence, \( M(1) = +\infty \).

We note also that these behave differently at the boundary of the radius of con
vergence, when \(|z| = 1\). In cases (a) and (b), the series diverge for all \( z \) with
\(|z| = 1\). (The terms don’t even go to zero.) In case (c), the series converges for
all values of \( z \) with \(|z| = 1\).

Thus you cannot say anything about a power series at its radius of convergence:
the behaviour can be arbitrarily wild.

**Remark 7**

I wrote the section on power series before the section on uniform convergence.
The arguments may seem quite repetitive. In fact, with the \( M \)-test in hand,
the discussion of power series could have been streamlined by using the \( M \)-test
with \( M_n = M(\rho)(r_1/\rho)^n \) for the last part of the proof of Proposition~ Radius of
convergence.

### 4.3 Hadamard’s n-th root formula

#### 4.3.1 Definition of limsup

Let \((a_n)\) be a sequence of real numbers. For each \( N \), define

\[ S_N = \{a_{N+1}, a_{N+2}, \ldots\}, \]

the set obtained by throwing away the first \( N \) terms in the sequence. Let

\[ M_N = \sup S_N, \]

possibly equal to \(+\infty\). Since \( S_N \subset S_{N-1} \), it follows that \( M_N \leq M_{N-1} \) for all
\( N \), so we have a non-increasing sequence of real numbers unless \( M_N = +\infty \) for
all \( N \).

Then

**Definition 8**
With the above definitions,

$$\limsup_{n \to \infty} a_n = \lim_{N \to \infty} M_N.$$ 

where $+\infty$ and $-\infty$ are both possibilities.

**Example 9**

If $a_n \to L$ as $n \to \infty$,

$$\lim_{n \to \infty} a_n = L.$$ 

**Example 10**

Suppose $a_n = 1 + 1/n$ if $n$ is even but $a_n = 0$ if $n$ is odd. Then

$$S_{2N-1} = \{1 + 1/(2N), -1, 1 + 1/(2N + 2), \ldots\}$$

The sup of this set is $1 + 1/2N$. The limit as $N \to +\infty$ is 1.

Intuitively, if $\limsup_{n \to \infty} a_n$ is finite, then it is the highest horizontal asymptote for the sequence (when graphed against $n$).

**Lemma 11**

Suppose that \((a_n)\) is a real sequence with

$$\limsup_{n \to \infty} a_n = L.$$ 

Then given any $\varepsilon > 0$ there exists $N_0 > 0$ such that which

$$n > N_0 \implies a_n < L + \varepsilon.$$ 

**Proof**

From the definition, there exists $N_0 > 0$ so that

$$N > N_0 \implies |M_N - L| < \varepsilon.$$ 

For such $N$, 

\[ \begin{align*}
\end{align*} \]
\[ L - \varepsilon < \sup S_N < L + \varepsilon. \]

and so \( a_n < L + \varepsilon \) for all \( n > N_0 \) as required.

**Remark 12**

The first inequality in Definition of \( \limsup \) shows that for each \( N > N_0 \) there exists \( n > N \) such that \( a_n > L - \varepsilon \). By taking an increasing sequence of \( N \), it follows that \( a_n > L - \varepsilon \) for infinitely many \( n \).

We are now ready to prove Hadamard’s ’formula’ for the radius of convergence of a complex power series.

**Theorem 4**

For the power series \( \text{https://en.wikitelearn.org/Course:Complex_Analysis_(Intermediate_Level)/Power_Series_Background/Radius_of_convergence} \), we have the formula

\[
\frac{1}{r} = \limsup_{n \to \infty} |a_n|^{1/n}
\]

for the radius of convergence.

**Proof**

Put

\[
\frac{1}{\rho} = \limsup_{n \to \infty} |a_n|^{1/n}
\]

Assume that \( \rho \neq 0, \infty \). Suppose first that \( s < r \), so

\[ |a_n|s^n \leq M(s) < \infty. \]

Taking \( n \) th roots,

\[ |a_n|^{1/n} \leq \frac{M(s)^{1/n}}{s}. \]

For any positive number \( C \), \( \lim_{n \to \infty} C^{1/n} \to 1 \) as \( n \to \infty \). By definition of the limit, it follows that for any given \( \varepsilon > 0 \), there exists \( N \) such that \( n > N \) implies \( C^{1/n} < 1 + \varepsilon \). Thus for \( n > N \), \( |a_n|^{1/n} < (1 + \varepsilon)/s \), and so

\[ \limsup_{n \to \infty} |a_n|^{1/n} \leq \frac{1 + \varepsilon}{s}. \]

This being true for every \( \varepsilon \), we have Hence \( 1/\rho \leq 1/s \) and \( s \leq \rho \). This is true for every \( s < r \), the radius of convergence, so finally
We aim to prove the opposite inequality. Suppose $\varepsilon > 0$ is much smaller than $\rho$. Then

$$\frac{1}{\rho} < \frac{1}{\rho - \varepsilon}$$

By the Lemma, we know that there exists $N_0$ so that $n > N_0$ implies that

$$|a_n|^{1/n} > \frac{1}{\rho - \varepsilon}.$$

If $s > \rho - \varepsilon$, this implies

$$|a_n|s^n > (\rho/(\rho - \varepsilon))^n \to +\infty \text{ as } n \to \infty.$$

This means that $M(s) = +\infty$ for all $s > \rho - \varepsilon$, and any $\varepsilon > 0$. This means that $r \geq \rho$. Combined with the previous inequality, we find that $r = \rho$. The cases $\rho = 0$ and $\rho = \infty$ require slight modifications of the argument, and are left as exercises.

**Remark 13**

The $n$ th root test has the advantage that it works always, provided one can calculate the limsup, of course. This in contrast to the ratio test, which only works if $\lim |a_{n+1}/a_n|$ exists.

**Proposition 14**

If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $r$, then the series $\sum_{n=0}^{\infty} na_n z^n$ has the same radius of convergence $r$.

**Proof**

This can be proved by more elementary means, but we shall prove it as an illustration of Hadamard’s formula.

We have observed in Remark Radius of convergence that the radius of convergence of

$$\sum na_n z^{n-1}$$

is the same as that of

$$\sum na_n z^n.$$
Now the $n$th root of the coefficient of $z^n$ in Definition of limsup is $n^{1/n}|a_n|^{1/n}$.
Since $\lim_{n \to \infty} n^{1/n} = 1$, it follows that

$$\limsup_{n \to \infty} n^{1/n}|a_n|^{1/n} = \limsup_{n \to \infty} |a_n|^{1/n} = 1/r.$$ 

By Hadamard’s formula, we conclude that the radius of convergence of

$r$.
Chapter 5

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