Course:Complex Analysis (Intermediate Level)/Functions in the complex plane/Integrability in the complex case

We introduce the theory of integrable functions in $\mathbb{C}$. From the integrability of a function it will follow that it is differentiable infinite times, where it is holomorphic.

We will not consider generic integrals, but *integrals along paths.*

Therefore, we have to define *curves* and *paths* in $\mathbb{C}$.

Definition

A *curve* in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}$ such that:

$$\forall\, t \in [a, b],\ \gamma(t) \in \mathbb{C}.$$

Definition

The set $\gamma^\star = \gamma([a, b]) = \{\gamma(t) \in \mathbb{C}\colon t \in [a, b]\}$ is edfined as the *support of the curve* γ . The points $\gamma(a)$ and $\gamma(b)$ are respectivelty the first and the second extremum of the curve. If $\gamma(a) = \gamma(b)$ we say that the curve is *closed.*"

Definition

A curve γ in $\mathbb{C}$ is a *path* if it is piecewise differentiable.

Now, we can give a meaning to the expression *integrate along a path*. The definition we will give is the same given in the course of Multivariable Calculus for the elements of an oriented path. All the remarks given in the case of a multivariable function follow: among them, we have the invariance of the integral with respect to the chosen oriented path. In the following, we can take $[a, b] \equiv [0, 1]$.

Definition

Let $\gamma : [a, b] \to \mathbb{C}$ be a path and $f : \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$ a complex-valued function. We define *integral of f along the path* γ the quantity:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Definition

We define the length of the curve (or the path) γ as the quantity:

$$L(\gamma) \equiv \int_0^1 \mid \gamma'(t) \mid dt.$$

We have that:

$$\mid \int_\gamma f(z)dz \mid \leq ML(\gamma), \text{ where} M \equiv max\{f(z) \colon z \in \gamma^\star \ .$$

An important concept, that distinguishes the integral along paths in $\mathbb{C}$ from the one in $\mathbb{R}^2$ is the *index*. In particular, we have the following:

Definition

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed path and $\Omega = \mathbb{C} \setminus \gamma^\star$. $\forall\, z \in \Omega$ we define *index of* z *with respect to* γ the quantity:

$$Ind_\gamma(z) = \frac{1}{2\pi i} \oint_\gamma \frac{d\xi}{\xi - z} \equiv \frac{1}{2\pi i} \oint dt \frac{\gamma'(t)}{\gamma(t) - z}.$$

In this defintion we go through γ in a counter-clockwise way. We set $Ind_\gamma(\infty) = 0$. We can see that every paths γ in $\mathbb{C}$ satisfy an important property, analogous to the one given by the *Jordan Theorem* for simple and closed curves in $\mathbb{R}^2$ .In fact, we have that $\gamma^\star$ always divides $\mathbb{C}$ in two connected components, one bounded and the other unbounded. There is an important relation between the function $Ind_\gamma(z)$ and the condition of holomorphism, as given by the following

Theorem

$Ind_\gamma(z)$ is an holomorphic function in $\mathbb{C}$. Moreover, we have that $Ind_\gamma \colon \Omega \subset \mathbb{C} \to \mathbb{Z}$, that is Ind_γ is an interger valued function and has constant valued on each connencted component in which $\mathbb{C}$ is splitted by γ . On the exterior of γ the index is always zero.

As for the definition of interior and exterior, we follow the *Jordan Theorem* for simple and closed curves in $\mathbb{R}^2$. We can refer to books of Multivariate Calucus about this topic.

We provide a proof of the previous theorem in a simple case:

Proof:

Let $\xi \in \mathbb{C}$ and γ be a circumference with centre z_0 and radius r . So, we have that $\gamma(t) = re^{it} + z_0$, $t \in [0, 2\pi]$. We obtain that:

$$Ind_\gamma(z) = \frac{1}{2\pi i} \oint_\gamma \frac{d\xi}{\xi - z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t) - z} dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = 1.$$

□

We remind the defintions of *connected, convex & simply connected* set.

Definition

We say that $\Omega \subset \mathbb{C}$ is connected if:
$\forall\, z_1, z_2 \in \Omega$ we have that $z(t) = (1-t)z_1 + tz_2 \subset \Omega$.

In general, for the other twoue definitions, we refer to textbooks of Multivariate Calculus, where one can find rigorous definitions for $\Omega \subset \mathbb{R}^n$, $n \geq 2$ that can be easily extended to $\mathbb{C}$.

We state two of the most important results in complex analysis, given the notion:

$$\hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}.$$

Theorem

We consider $\Omega \subset \hat{\mathbb{C}}$ and f an holomorphic function in Ω. If γ is a closed path in Ω such that $Ind_\gamma(z) = 0$, $\forall\, z \in \hat{\mathbb{C}} \setminus \Omega$, then:

$$\int_\gamma f(z)dz = 0.$$

Equivalently, we have that:

Let $\Omega \subset \hat{\mathbb{C}}$ be a *simply connected* set (and so $\hat{\mathbb{C}} \setminus \Omega$ is connected) and f be a holomorphic function in Ω. If γ is a closed path in Ω , then:

$$\int_\gamma f(z)dz = 0.$$

We see that the theorem provides a *necessary condition* for the integral along the path to be zero, in fact we could have cases in which the curve has a non-zero index and TEST so we could not say anything about the value of the integral a priori.

An important corollary of Cauchy theorem gives conditions for the comparison between integrals of the same holomorphic f along two different closed paths in Ω .

Theorem

We consider $\Omega \subset \hat{\mathbb{C}}$ and f an holomorphic function in Ω. Let γ_1 & γ_2 be two closed paths in Ω such that $Ind_{\gamma_1}(z) = Ind_{\gamma_2}(z)$, $\forall\, z \in \hat{\mathbb{C}} \setminus \Omega$, then:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof:

Let f be holomorphic in Ω and γ, λ be two closed paths. We link γ and λ with two line segments $\pm c$ with a distance ϵ between them. It is not possible to have a closed path with zero index with respect to every $z \in \hat{\mathbb{C}} \setminus \Omega$. Let such Γ given by: $\Gamma = \gamma \,\cup\, c \,\cup\, -\lambda \,\cup\, -c$. By applying Cauchy's theorem to the function f considered:

$$0 = \int_{\Gamma} f(z)dz = \int_{\gamma} f(z)dz + \int_{c} f(z)dz + \int_{(-\lambda)} f(z)dz + \int_{(-c)} f(z)dz =$$
$$= \int_{\gamma} f(z)dz + \int_{c} f(z)dz - \int_{\lambda} f(z)dz - \int_{c} f(z)dz =$$
$$= \int_{\gamma} f(z)dz - \int_{\lambda} f(z)dz.$$

It follows that: $\int_{\gamma} f(z)dz = \int_{\lambda} f(z)dz$.

□

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1.1 Text

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