Course:Statistical Mechanics/The Renormalization Group/The origins of scaling and critical behaviour

Let us consider a fixed point of the RG flow of a generic system, and assume that it has two relevant directions corresponding to the coupling constants T^*[1], the temperature, and H , the external field. We suppose that T and H are transformed under the RG as:

$$T' = R^T_\ell(T,H) \qquad H' = R^H_\ell(T,H)$$

where R^T_ℓ and R^H_ℓ are analytic functions given by the coarse graining procedure. The fixed points (T^*,H^*) of the flow will be given by the solutions of:

$$T^* = R^T_\ell(T^*,H^*) \qquad H^* = R^H_\ell(T^*,H^*)$$

Linearising the transformation around (T^*,H^*) , in terms of the reduced variables $t=(T-T^*)/T^*$ and $h=(H-H^*)/H^*$ we have:

$$\begin{pmatrix} t' \\ h' \end{pmatrix} = \boldsymbol{T}\begin{pmatrix} t \\ h \end{pmatrix}$$

where:

$$\boldsymbol{T} = \begin{pmatrix} \partial R^T_\ell/\partial T & \partial R^T_\ell/\partial H \\ \partial R^H_\ell/\partial T & \partial R^H_\ell/\partial H \end{pmatrix}_{|T^*,H^*}$$

As previously stated we suppose $\boldsymbol{T}$ to be diagonalizable. We therefore write its eigenvalues as:

$$\lambda^t_\ell = \ell^{y_t} \qquad \lambda^h_\ell = \ell^{y_h}$$

Note that we can always do that, it is just a simple definition. In other words, we are defining y_t and y_h as:

$$y_t = \frac{\ln \lambda^t_\ell}{\ln \ell} \qquad y_h = \frac{\ln \lambda^h_\ell}{\ln \ell}$$

This way we can write:

$$\begin{pmatrix} t' \\ h' \end{pmatrix} = \begin{pmatrix} \lambda_\ell^t & 0 \\ 0 & \lambda_\ell^h \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} t' \\ h' \end{pmatrix} = \begin{pmatrix} \ell^{y_t} t \\ \ell^{y_h} h \end{pmatrix}$$

After n iterations we will have:

$$t^{(n)} = (\ell^{y_t})^n \, t \qquad h^{(n)} = (\ell^{y_h})^n \, h$$

and since ingeneral we know that $\xi(t', h') = \xi(t, h)/\ell$:

$$\xi(t, h) = \ell^n \xi(\ell^{n y_t} t, \ell^{n y_h} h)$$

This is the scaling law of the correlation length. From this we can determine the critical exponent ν ; in fact, setting $h = 0$ and choosing ℓ so that $t \ell^{n y_t} = b$ with b a positive real number*[2], we have:

$$\ell^n = \left(\frac{b}{t} \right)^{1/y_t} \quad \Rightarrow \quad \xi(t) = \left(\frac{t}{b} \right)^{-1/y_t} \xi(b, 0)$$

Since in general $\xi \sim t^{-\nu}$, we get:

$$\nu = \frac{1}{y_t}$$

This is an extremely important result! In fact, we see that once the RG transformation R_ℓ is known, y_t is straightforward to compute and so we are actually able to calculate ν and predict its value! We can do even something more (including giving y_h a meaning) from the scaling law of the free energy density. After n iterations of the RG we have:

$$f(t, h) = \ell^{-nd} f(t^{(n)}, h^{(n)}) = \ell^{-nd} f(\ell^{n y_t} t, \ell^{n y_h} h)$$

and choosing ℓ so that $\ell^{n y_t} t = b^{y_t}$, then:

$$f(t, h) = t^{d/y_t} b^{-d} f(b^{y_t}, b^{y_h} h / t^{y_h/y_t})$$

Comparing this to what we have seen in An alternative expression for the scaling hypothesis we get:

$$2 - \alpha = \frac{d}{y_t} \qquad \Delta = \frac{y_h}{y_t}$$

[1] We have already stated that considering K as a coupling constant is equivalent to considering T as such.

[2] Remember that the value of ℓ is not fixed, so we can choose the one we prefer; in this case we are making this choice because ℓ does not necessarily have to be an integer.

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1.1 Text

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