It is a very rare fact that a model of interacting degrees of freedom can be exactly solved\[1\], and generally a model is not solvable in any dimension (think about the Ising model) so we must find other ways to study such systems in order to understand the possible occurrence of phase transitions and their behaviour near possible critical points. The most simple method, and the first to which one usually resorts to, is the so called mean field approximation. One common feature of mean field theories is the identification of an appropriate order parameter; then, in very general and abstract words, there are two different approaches that can be taken (we shall see both):

- approximating an interacting system by a non-interacting one in a self-consistent external field expressed in terms of the order parameter
- expressing an approximate free energy in terms of this parameter and minimize the free energy with respect to the order parameter

In other (and maybe clearer) words, the first approach consists in substituting a system of interacting degrees of freedom with another system where these degrees of freedom do not interact but are subject to the action of an external mean field, which approximates the action of all the degrees of freedom on a single one; the second approach on the other hand is an “extension” in statistical mechanics of variational principles.

In order to be a little bit more explicit, let us see how the first approach applies to the Ising model\[2\]. In the case of nearest-neighbour interactions, the reduced Hamiltonian of the system is:

$$-\beta \mathcal{H} = K \sum_{(i,j)} S_i S_j + h \sum_i S_i$$

Consider now the single spin $S_i$ ; we can note that the first term of the reduced Hamiltonian can be written as:

$$K \sum_{(i,j)} S_i S_j = -K \sum_i S_i \hat{h}_i(\{S\})$$

where:
\[
\hat{h}_i(\{S\}) = \sum_{j \in \text{n.n.}(i)} S_j
\]

where with the notation \( j \in \text{n.n.}(i) \) we mean that \( j \) is a nearest neighbour of \( i \). We thus see that every spin \( S_i \) is subjected to an internal field \( \hat{h}_i \) due to the presence of the nearest neighbouring spins. Now, if the number \( z \) of nearest neighbours is large (which happens in high dimension, or equivalently we could assume interactions with a larger number of spins, for example including also the so called next nearest neighbours) this internal field can be approximated with the mean field generated by all the other spins in the lattice:

\[
\frac{1}{z} \sum_{j \in \text{n.n.}(i)} S_j \approx \frac{1}{N} \sum_j S_j
\]

This suggests that can be rewritten as:

\[
-K \sum_i S_i \sum_j S_j \frac{z}{N}
\]

If we consider an Ising model with an hypercubic lattice and simple nearest-neighbour interactions, we know that \( z = 2d \) so the mean field term can be written as:

\[
\hat{h}_{\text{m.f.}} = 2 \frac{dK}{N} \left( \sum_i S_i \right)^2
\]

since the two sums are independent. We therefore have that the approximation (with \( z = 2d \)) is better the larger the dimensionality \( d \) of the system is. From this very simple observation we can argue something that will become explicit and clear only later on (see Introduction: Ginzburg criterion): mean field theories are good approximations only if the dimensionality of the system is large enough.

Let us now come back to the general properties of mean field theories. One of the main features of mean field theories is that they neglect the effects of fluctuations in the order parameter (in other words, within mean field theories the order parameter is supposed to be constant over all the system): on one hand we will see that this will make it possible to study a lot of systems, and to obtain loads of interesting and useful information about them, but on the other one we will see that this will be fatal for the reliability of mean field theories in the proximity of critical points, since they are characterised by the divergence of long-ranged fluctuations (see Long range correlations). This also means that mean field theories are in any case rather efficient far from critical points.

There are many possible mean field theories, and we are mainly going to study them applied to the Ising model (since it’s the one we know better until now) or similar ones.

[1] An exhaustive review of exactly solvable models is given in .

[2] We now just make some qualitative observations; the mean field theory for the Ising model will be treated in detail shortly.
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