Course:Statistical Mechanics/Scaling theory/Kadanoff's argument for the Ising model

Let us consider a d -dimensional Ising model with hypercubic lattice with lattice constant a ; assuming nearest-neighbour interactions the Hamiltonian of the system will be:

$$-\beta\mathcal{H} = K\sum_{\langle i,j\rangle} S_i S_j + h\sum_i S_i$$

where $K = \beta J$ and $h = \beta H$, as usual.

Since the values of the spin variables are correlated on lengths of the order of $\xi(T)$, the spins contained in regions of linear dimension ℓa , with ℓ such that:

$$a \ll \ell a \ll \xi(T)$$

will behave, statistically, as a single unit. We can therefore imagine to carry out, similarly to what we have seen for the Ginzburg-Landau theory (see Functional partition function and coarse graining), a coarse graining procedure were we substitute the spin variables S_i inside a "block" of linear dimension ℓa (which will therefore contain ℓ^d spins) with a single *block spin*; the total number of blocks will of course be $N_{\text{b}} = N/\ell^d$. Considering the I -th block, we can define the block spin S_I as:

$$S_I = \frac{1}{|m_\ell|}\cdot\frac{1}{\ell^d}\sum_{i\in I} S_i$$

where the mean magnetization of the I -th block m_ℓ is:

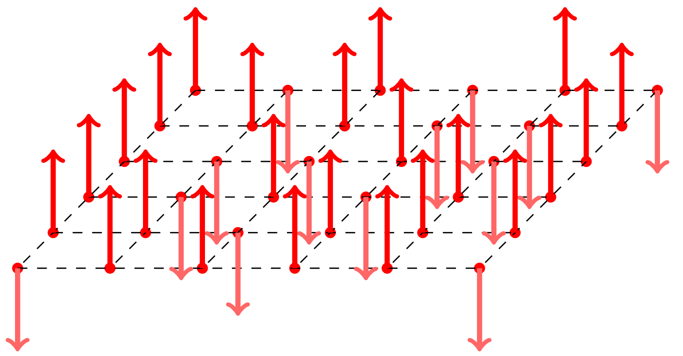
$$m_\ell = \frac{1}{\ell^d}\sum_{i\in I}\langle S_i\rangle$$

This way, the new block spin variables can assume only the values ± 1 , just like the original ones.

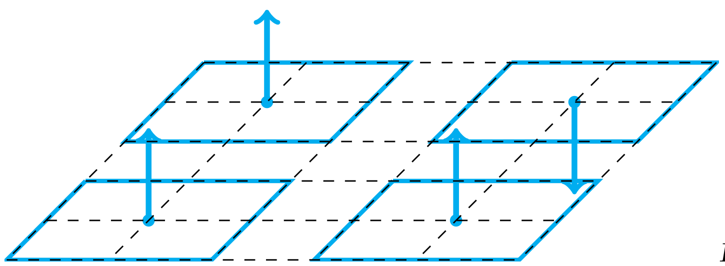
In the end we are left with a system of block spins on a hypercubic lattice with lattice constant ℓa . We can therefore *rescale* the spatial distances between the degrees of freedom of our system:

$$\vec{r}_\ell = \frac{\vec{r}}{\ell}$$

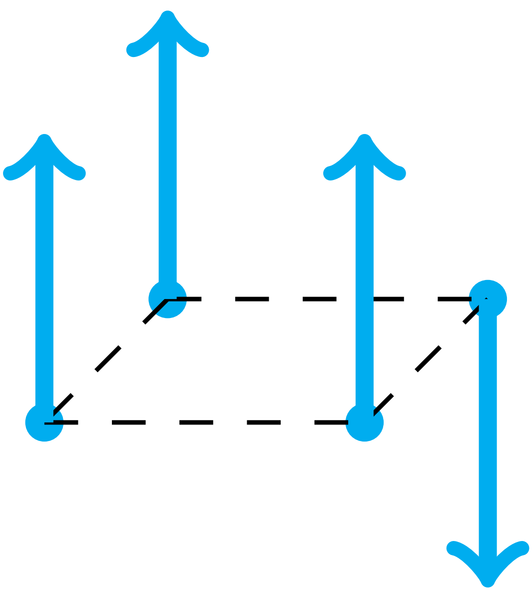
In other words, since ℓa is now the characteristic length of the system we are measuring the distances in units of ℓa (just like in the original one we measured distances in units of a). The coarse graining procedure we have just seen is described by the following figures for a two-dimensional Ising model:



Original system



Block spins



Final rescaled system

Kadanoff's argument now proceeds with two assumptions. The first one states that, in analogy to what happens in the original system, we assume that the block spins interact with the nearest neighbours and an external effective field (just like the original ones do).

This means that the Hamiltonian of the coarse grained system of block spins has the same form of the original one, of course provided that the spins, coupling constants and external fields are redefined. If we call K_ℓ and h_ℓ these new constants (of course $K_1 = K$ and $h_1 = h$) the new effective Hamiltonian is:

$$-\beta\mathcal{H}_\ell = K_\ell \sum_{\langle I,J\rangle} S_I S_J + h_\ell \sum_{I=1}^{N/\ell^d} S_I$$

Since in the new system the lengths have been rescaled by a factor ℓ , as in , this means that also the correlation length of the system has to be measured in units of ℓa , and in particular we will have:

$$\xi_\ell = \frac{\xi}{\ell}$$

This means that the new system has a *lower* correlation length, and it will thus be *farther* from the critical point with respect to the original one, and so will have a new effective temperature t_ℓ . Similarly, in the coarse grained system the magnetic field will be rescaled to an effective one:

$$h\sum_i S_i = h\sum_I \sum_{i\in I} S_i = hm_\ell \ell^d \sum_I S_I := h_\ell \sum_I S_I$$

which implies that there is a relation between the new magnetic field and the mean magnetization:

$$h_\ell = hm_\ell \ell^d$$

Since the Hamiltonian of the block spin system has the same form of the original one, the same will be true also for the free energy, provided that h , K and N are substituted with h_ℓ , K_ℓ and N/ℓ^d ; in particular, considering the singular part f_s of the free energy density we will have:

$$\frac{N}{\ell^d} f_\mathrm{s}(t_\ell, h_\ell) = N f_\mathrm{s}(t, h)$$

and so:

$$f_\mathrm{s}(t_\ell, h_\ell) = \ell^d f_\mathrm{s}(t, h)$$

In order to proceed, we now need the second assumption. We assume that:

$$t_\ell = t\ell^{y_t} \qquad h_\ell = h\ell^{y_h} \qquad y_t, y_h > 0$$

The justification of this assumption lies in the fact that we are trying to understand the scaling properties of our system near a critical point, and these are the simplest possible relations between (t, h) and (t_ℓ, h_ℓ) that satisfy the following symmetry requirements:

- when $h \to -h$, then $h_\ell \to -h_\ell$
- when $h \to -h$, then $t_\ell \to t_\ell$
- when $t = h = 0$, then $t_\ell = h_\ell = 0$

The exponents y_t and y_h are for now unspecified, apart from the fact that they must be positive (so that the coarse grained system is indeed farther from the critical point with respect to the original one).

If we use this assumption in we get:

$$f_\mathrm{s}(t,h) = \frac{1}{\ell^d} f_\mathrm{s}(t\ell^{y_t}, h\ell^{y_h})$$

This is very similar to Widom's scaling hypothesis, where the parameter λ is the inverse of the block volume ℓ^d . Since ℓ has no specified value, we can choose the one we want and again we use the properties of generalized homogeneous functions to eliminate one of the arguments of f_s . In particular, setting $\ell = |t|^{-1/y_t}$ we get:

$$f_\mathrm{s}(t,h) = |t|^{d/y_t} f_\mathrm{s}\left(1, h|t|^{-y_h/y_t}\right)$$

where the gap exponent is now $\Delta = y_h/y_t$; comparing this equation with the alternative expression of the scaling hypothesis we have:

$$2 - \alpha = \frac{d}{y_t}$$

This relation will be used in the following, after we have discussed the scaling of correlation functions.

1 Kadanoff's scaling for correlation functions

Let us now consider the correlation function of the block spin system:

$$G(\vec{r}_\ell, t_\ell) = \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$$

where $\vec{r}_\ell$ is the vector of the relative distance between the centers of the I -th and J -th block (measured in units of ℓa , as stated before). We want now to see how this correlation length is related to the one of the original system $G(\vec{r}, t)$. From we have $m_\ell = h_\ell \ell^{-d}/h = \ell^{y_h - d}$, and so we get:

$$G(\vec{r}_\ell, t_\ell) = \langle S_I S_J \rangle - \langle S_I \rangle \rangle S_J \rangle = \frac{1}{\ell^{2(y_h - d)} \cdot \ell^{2d}} \sum_{i \in I} \sum_{j \in J} \left(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \right) =$$

$$= \frac{\ell^{2d}}{\ell^{2(y_h - d)} \cdot \ell^{2d}} \left(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \right) = \ell^{2(d - y_h)} G(\vec{r}, t)$$

Introducing also the dependence on h , we have:

$$G\left(\frac{\vec{r}}{\ell}, t\ell^{y_t}, h\ell^{y_h}\right) = \ell^{2(d-y_h)}G(\vec{r}, t, h)$$

which is an equivalent of Widom's assumption for the correlation function. Again, we can remove the dependence on t setting $\ell = t^{-1/y_t}$ so that:

$$G(\vec{r}, t, h) = t^{2\frac{d-y_h}{y_t}} G\left(\vec{r}t^{1/y_t}, 1, ht^{-y_h/y_t}\right)$$

Now, $\vec{r}$ scales with ℓ as all the lengths of our system, and since we have set $\ell = t^{-1/y_t}$ we have $rt^{1/y_t} = 1$. Therefore:

$$G\left(\vec{r}t^{1/y_t}, 1, ht^{-y_h/y_t}\right) = \left(rt^{1/y_t}\right)^{-2(d-y_h)} F_G\left(\vec{r}t^{1/y_t}, ht^{-y_h/y_t}\right)$$

where in the last step we have defined $F_G(a, b) = G(a, b, 1)$. Therefore:

$$G(\vec{r}, t, h) = \frac{1}{r^{2(d-y_t)}} F_G\left(\vec{r}t^{1/y_t}, ht^{-y_h/y_t}\right)$$

With the choice $\ell = t^{-1/y_t}$ we further have that the correlation length scales as:

$$\xi = \ell\xi_\ell = \xi_\ell t^{-1/y_t}$$

so we also have:

$$\nu = \frac{1}{y_t}$$

This equation together with

$$2 - \alpha = \frac{d}{y_t}$$

leads to the *hyperscaling relation*:

$$2 - \alpha = \nu d$$

Hyperscaling relations are known to be less robust than the normal scaling relations between critical exponents (for example, for Hamiltonians with long-ranged power law interactions hyperscaling relations don't hold).

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2.1 Text

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