Course: Statistical Mechanics/Statistical mechanics of phase transitions/Absence of spontaneous magnetization

Let us now come back to free boundary conditions and compute the mean magnetization $\langle S_j \rangle$ of a given site $j$. By definition we have $^1$:

$$Z \langle S_j \rangle = \text{Tr} \left( S_j e^{-\beta H} \right) = \sum_{\{S_i = \pm 1\}} S_j e^{K \sum_{i=1}^{N-1} S_i S_{i+1}} = \sum_{\{S_i = \pm 1\}} \prod_{i=1}^{N-1} S_j e^{K S_i S_{i+1}}$$

We now use the fact that $e^x = \cosh x + \sinh x$, so that in this case:

$$e^{K S_i S_{i+1}} = \cosh K + S_i S_{i+1} \sinh K$$

where we have used (like we have done previously) the evenness of $\cosh$, and now also the oddness of $\sinh$, i.e. $\sinh(\pm x) = \pm \sinh x$. This way:

$$Z \langle S_j \rangle = \sum_{\{S_i = \pm 1\}} \prod_{i=1}^{N-1} S_j (\cosh K + S_i S_{i+1} \sinh K) =$$

$$(\cosh K)^{N-1} \sum_{\{S_i = \pm 1\}} \prod_{i=1}^{N-1} S_j (1 + S_i S_{i+1} \tanh K) =$$

$$(\cosh K)^{N-1} \sum_{S_1 = \pm 1} \cdots \sum_{S_N = \pm 1} S_j (1 + S_1 S_2 \tanh K)(1 + S_2 S_3 \tanh K) \cdots (1 + S_{N-1} S_N \tanh K) =$$

$$(\cosh K)^{N-1} \sum_{S_1 = \pm 1} \cdots \sum_{S_N = \pm 1} \left[ S_j + \sum_{M=1}^{N-1} S_j S_{i_1} S_{i_1+1} S_{i_2} S_{i_2+1} \cdots S_{i_M} S_{i_M+1} (\tanh K)^M \right]$$

where $S_{i_1}$ etc. are the spin variables appropriately rearranged $^2$. Let us now consider the two different contributions to $\langle S_j \rangle$. As of the first:
\[ \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} S_j = 2^{N-1} \sum_{S_j=\pm 1} S_j = 2^{N-1}(1-1) = 0 \]

Considering now the second one:

\[ \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \sum_{M=1}^{N-1} S_j S_{i_1} S_{i_1+1} S_{i_2} S_{i_2+1} \cdots S_{i_M} S_{i_M+1} (\tanh K)^M \]

we have that for every fixed \( M \) this term vanishes; for example, if we consider the contribution relative to a fixed value \( M^* \) of \( M \) and sum (for example) over \( S_{i_1} \) we have:

\[
S_j \cdot (+1) \cdot S_{i_1+1} S_{i_2} S_{i_2+1} \cdots S_{i_{M^*}} S_{i_{M^*}+1} (\tanh K)^{M^*} + \\
+ S_j \cdot (-1) \cdot S_{i_1+1} S_{i_2} S_{i_2+1} \cdots S_{i_{M^*}} S_{i_{M^*}+1} (\tanh K)^{M^*} = 0
\]

Therefore, also the second term vanishes and in the end:

\[ \langle S_j \rangle = 0 \]

This result perfectly agrees with what we have already seen in Absence of phase transitions for finite systems, but now it has been deduced from a direct computation.

1. Note that the sum on nearest neighbour is done without counting the same terms twice (as we have already stressed). In fact, in our case every spin \( S_i \) interacts with its nearest neighbours \( S_{i-1} \) and \( S_{i+1} \), but the sum in the trace involves every two-spin interaction only once.

2. The form of this last term can be understood more easily doing an explicit computation with a simple example.
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