We specialise slightly and consider $p : E \rightarrow M$, a complex vector bundle of rank $r$, with base space $M$ a compact (orientable) smooth manifold of real dimension $n$.

A **generic section** is a section of the bundle which intersects the zero-section $\mathcal{O}_M$ transversally. Two sections are said to have **transversal intersection** if at every point of intersection, the tangent spaces of the sections at the point generate the tangent space of the ambient space at the point.

We can define the Chern classes in terms of the zero sets of generic sections, namely the transversal intersections of the generic sections with $\mathcal{O}_M$.

For a generic section $s$, we write $Z(s)$ for the zero-set of $s$. Since $s$ is transverse to the zero section, $Z(s)$ is a submanifold of $M$ of real codimension $2r$. Applying Poincaré duality to the fundamental class $[Z(s)] \in H_{n-2r}(M)$, we obtain a cohomology class in $H^{2r}(M)$. This is called the **Euler class** $e(E)$ of the vector bundle and is also the $r$th **Chern class**.

$$e(E) = c_r(E) := [Z(s)] \in H_{n-2r}(M) \cong H^{2r}(M; \mathbb{Z})$$

By analogy we extend the definition to give us a sequence of cohomology classes:

$$c_i(E) := PD([Z(s_1 \wedge \ldots \wedge s_{r-i+1}])] \in H^{2i}(M; \mathbb{Z})$$

for $s_1, \ldots, s_{r-i+1}$ generic sections. Here $Z(s_1 \wedge \ldots \wedge s_{r-i+1})$ can be viewed as the set of points $m \in M$ where the sections $s_1(m), \ldots, s_{r-i+1}(m)$ become linearly dependent. Note then that

$$c_1(E) := PD([Z(s_1 \wedge \ldots \wedge s_r)]) H^{2r}(M; \mathbb{Z})$$

We define $c_0(E) = 1$ and call the sum $c(E) = 1 + c_1(E) + c_2(E) + \ldots \in H^*(M; \mathbb{Z})$ the **total Chern class** of $E$. Point (3) in Theorem 4.2 below demonstrates that such a sum is finite, i.e. a well-defined element of $H^*(M; \mathbb{Z})$.

**Theorem 4.2**
There is a unique sequence of functions $c_1, c_2, \ldots$ assigning to each complex vector bundle $E \to B$ a class $c_i(E) \in H^{2i}(B, \mathbb{Z})$ depending only on the isomorphism type of $E$ and satisfying:

1. $c_i(f^*E) = f^*(c_i(E))$ for a pullback $f^*E$
2. $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$
3. $c_i(E) = 0$ if $i > \text{rank} E$
4. For the canonical line bundle $E \to \mathbb{C}P^\infty$, $c_1(E)$ generates $H^2(\mathbb{C}P^\infty; \mathbb{Z})$. 

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